ON THE TRANSONIC FLOW AROUND WING PROFILES AND REVOLUTION BODIES AT ZERO ANGLE OF ATTACK

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INTRODUCTION

THE problem studied here is essentially one of determining the influence of the unsteadiness on the transonic flow of a uniform and subsonic stream past an obstacle which at a given instant starts moving with a uniformly accelerated translatory motion relative to the undisturbed flow. We assume $K \equiv 0(\delta^{2/3})$; $1-M_\infty^2 \equiv 0(\delta^{2/3})$ K being the parameter of unsteadiness $K = \left(\frac{Al}{q_\infty^2}\right)^{1/2}$; A is the acceleration of the body; l a characteristic length; q_∞ the velocity of the flow at infinity; δ maximum thickness of the body, or another equivalent parameter.

We first examine the flow around a simple wedge profile for which a sufficiently approximate solution may be obtained.

Considering then any profile whatever and assuming that in the initial configuration of the flow there are shock waves starting from the contour of said profile, we seek the influence of K on the shape of these waves and their propagation velocity. To answer the first of these two questions, we try to construct local solutions of the equation of motion around the point A from which the wave is detached for each of the two parts in which the front of the latter divides the field, which join on the wave line.

In order to evaluate, at least qualitatively, the influence of K on the propagation speed of the wave, we consider the limiting case when the profile suddenly varies its velocity with respect to the fluid at the infinite ($K \to \infty$).

The quantitative determination of the unsteady flow with shock waves is a problem of extreme difficulty; these difficulties become attenuated if the Mach number is close to one, since in these conditions the shock waves are fixed to the trailing edge, and the variation of their shape with the time hardly influences the configuration of the upstream field, and therefore the coefficients of pressure C_p on the contour, for whose calculation it is possible to employ approximate methods; particularly useful to this purpose, is the method of J. R. Spreiter and A. Y. Alskne[1], and here we will indicate how this method can be extended to the study of the non-stationary transonic motions around wing profiles and revolution body.

1. LIST OF SYMBOLS

a = sound velocity; a^* = critical velocity; a_r^* = critical velocity of the relative stream

A = acceleration of the translatory motion of the profile

C_p = pressure coefficient $= \dfrac{p-p_\infty}{\frac{1}{2}\varrho_\infty q_\infty^2}$

c_r = drag coefficient

$D = \dfrac{\delta^{2/3}}{1-M_\infty}$

$f(x, y, \tau)$ = reduced potential of the perturbed flow relative to the velocity at the infinite $= (\gamma+1)^{1/3}(\delta/M_\infty)^{-2/3}\phi$

$f^*(x, y, \tau)$ = reduced potential of the perturbed flow relative to the critical velocity

f^{**} = reduced potential of the perturbed flow in the coordinates (X, Y, τ)

$h = \dfrac{2K}{\delta^{2/3}(1+\gamma)^{2/3}}$; $\quad K = \left(\dfrac{Al}{q_\infty^2}\right)^{1/2}$; $\quad K^* = h/r$

l = length of the chord

M = Mach number (local); M_∞ = Mach number of the flow at the infinite

p = pressure p_∞ = pressure of the flow at the infinite

q = velocity q_∞ = velocity of the flow at the infinite

R_c = radius of curvature of the profile

I = curvilinear abscissa measured along the shock wave

t^* = time; $\quad t = \dfrac{t^* q_\infty}{l}$

u, v = velocity components; they have different meanings depending on the sections; $u^* = -u$; $v_0 = 1+\dfrac{2}{\gamma+1}(1-M_\infty)$;

$$u_\infty^* = \frac{2(1-M_\infty)}{(\gamma+1)^{2/3}(\delta/M_\infty\varphi)^{2/3}}$$

x^{α}, y^{x} = coordinates of the absolute reference system (conventionally) in the physical plane (x^{α} in the direction of the axis of symmetry of the profile);

$$x = \frac{x^{\alpha}}{l}; \quad y = \frac{y^{x}}{l}; \quad Y = M_{\infty}^{2/3}\delta^{1/3}(1+\gamma)^{1/3}y$$

X = abscissa with respect to a system of axes fixed to the profile $= x + \frac{1}{2}\tau^2$

X_a, Y_a = coordinates of the points of the profile from which the shock wave starts

V_0 = velocity of propagation of the shock wave with respect to the profile

$V_{0,a}$ = velocity of propagation of the shock wave with respect to the system (x, y)

$$z = \frac{2}{3}u^{*3/2}; \quad z_{\infty} = \frac{2}{3}u_{\infty}^{*3/2}$$

γ = specific heat ratio

δ^* = maximum thickness of the profile; $\delta = \dfrac{\delta^*}{l}$

$q_{\infty}\varepsilon$ = excess of velocity relative to the critical velocity (in the flow relative to the profile) in the point the shock wave starts from

ξ, μ = axes with their origin in the point from which the shock wave starts (ξ = tangent to the profile; μ normal); $\zeta = \xi + i\mu$

$lq_{\infty}\phi^*$ = potential of the flow (with respect to the x,y axes)

$lq_{\infty}\phi$ = potential of the perturbed flow relative to the flow at the infinite

$lq_{\infty}\bar{\phi}^*$ = potential of the flow relative to the profile

$lq_{\infty}\bar{\phi}$ = perturbed potential of the relative stream with respect to the critical velocity

$\bar{\bar{\phi}}$ = potential as defined by (27)

The meaning of other symbols is indicated in the context.

2. FLOW AROUND THE WEDGE PROFILE. EQUATION OF MOTION. BOUNDARY CONDITIONS

Let us relate the motion to a system of axes (x^*, y^*) which we consider as fixed and relative to which the flow at the infinite has the velocity $\vec{q}_{\infty}$, directed along (x^*) (Fig. 1), whereas the wedge moves with a uniformly accelerated translatory motion with velocity, $\vec{q}^1 = -At^*\vec{i}$: f, being the potential of the perturbed flow as defined in section (1), and f,x,y,τ the space and time coordinates expressed as a function of x^*, y^*, and τ^* by the relations given in section (1), we may write

$$\frac{K^2}{(\gamma+1)^{2/3}\delta^{2/3}}f_{\tau\tau}+\frac{2K}{(\gamma+1)^{2/3}\delta^{2/3}}f_{x,\tau}+\left[\frac{M_\infty^2-1}{M_\infty^2(\gamma+1)^{2/3}\delta^{2/3}}+\frac{f_x}{M_\infty^{2/3}}\right]f_{xx}-f_{YY}\frac{1}{M_\infty^{2/3}}=0 \quad (1)$$

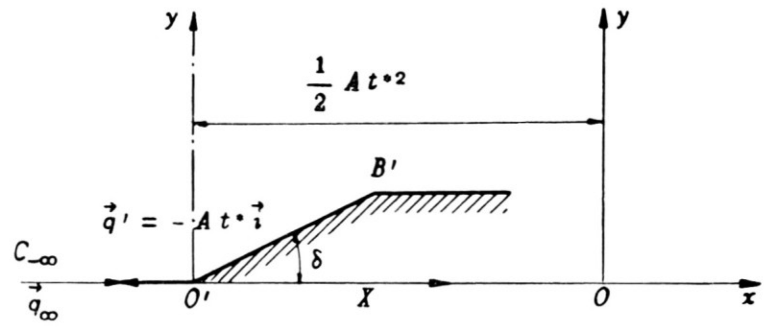

FIG. 1.

wherein the meaning of the various symbols is given in section (1); (1) is the equation of the "small perturbations" in the assumption that

$$(1-M_\infty)\equiv 0(\delta^{2/3}); \qquad K\equiv 0(1-M_\infty)=0(\delta^{2/3}) \quad (2)$$

and it differs from that given by C. C. Lin, E. Reissner and H. S. Tsien[2] due to the presence of the term in K^2, since it is not permissible to neglect the term in the second derivative with respect to τ of f, even if K is small, of the order given by (2), for values of τ/K in the order of one, or for values of x in the order of $(1/K)$.

We assume now that

$$\left.\begin{aligned} f^* &= f-(\gamma+1)^{1/3}\left(\frac{\delta}{M_\infty}\right)^{-2/3}\left(\frac{a^*}{q_\infty}-1\right)x \\ &= f-(\gamma+1)^{2/3}\left(\frac{\delta}{M_\infty}\right)^{-2/3}2(1-M_\infty)x \\ \delta^{2/3} &= D(1-M_\infty); \qquad \frac{K}{(\gamma+1)^{2/3}D}=K^* \end{aligned}\right\} \quad (3)$$

Thus, (1) becomes

$$-KK^*f^*_{\tau\tau}-2K^*f^*_{\tau x}-f^*_xf^*_{xx}+f^*_{yy}=0 \quad (4)$$

The corresponding conditions at the contour are, x_0 being the abscissa of the wedge apex 0′ for $\tau=0$:

for

$$iY=0_i \begin{cases} -\infty \leqslant x \leqslant x_0-\frac{\tau^2}{2}, \quad f_y^*=0 \\ x_0-\frac{\tau^2}{2} \leqslant x \leqslant 1+x_0\frac{\tau^2}{2} \\ f_y^*=1+\frac{2}{\gamma+1}(1-M_\infty)+K\tau+\frac{1}{\gamma+1}\frac{K}{K^*}f_x^* \\ 1+x_0\frac{\tau^2}{2} \leqslant x \leqslant \infty, \quad f_y^*=8 \end{cases} \tag{5}$$

for $\tau=0$, the potential must reduce itself to the perturbed potential corresponding to the uniform steady flow;
for

$$x^2+y^2 \to \infty, \quad \begin{cases} f_x^* = -(\gamma+1)^{1/3}\left(\frac{\delta}{M_\infty}\right)^{-2/3}\frac{2}{\gamma+1}(-M_\infty)=\mu_\infty \\ f_y^*=0 \end{cases} \tag{6}$$

for whatever τ.

The form of equation (4) induces us to assume, for K^* enough smaller than one,

$$f^*=f_0^*+K^*f_1^*+K^{+2}f_2^*+\cdots \tag{7}$$

wherein the equations fulfilled by the f_n^* are obtained by substituting (7) in (4) and equating to zero coefficients of the successive K^*. Thus, we obtain

$$-f_{0,x}^*f_{0,xx}^*+f_{0,yy}^*=0 \tag{8}$$

on the conditions

$$\left.\begin{aligned} &f_{0,y}^*=0, \quad \text{for} \quad Y=0; \quad -\infty \leqslant x \leqslant x_0-\frac{\tau^2}{2} \\ &f_{0,Y}^*=1+\frac{2}{\gamma+1}(1-M_\infty)+\frac{1}{\gamma+1}\frac{K}{K^*}f_{0,x}^* \\ &\qquad \text{for} \quad Y=0; \quad \mu_0-\frac{\tau^2}{2} \leqslant x \leqslant 1+x_0-\frac{\tau^2}{2} \\ &f_{0,y}^*=0 \quad \text{for} \quad Y=0; \quad 1+x_0-\frac{\tau^2}{2} \leqslant u \leqslant \infty \end{aligned}\right\} \tag{8'}$$

$$\left.\begin{aligned} &f_{0,\tau x}^*=f_{0,\tau y}^*= \quad \text{for} \quad \tau=0; \\ &f_{0,x}^*=-(\gamma+1)^{1/3}\left(\frac{\delta}{M_\infty}\right)^{-2/3}\frac{2}{\gamma+1}(1-M_\infty)=u_\infty; \quad f_{0,Y}^*=0 \\ &\qquad \text{for} \quad x^2+y^2 \to \infty \end{aligned}\right\} \tag{8''}$$

$$-f_{0,x}^*f_{1,xx}^*-f_{0,xx}^*+f_{1,yy}^*=2f_{0,x\tau}^* \tag{9}$$

on the conditions:

$$\left.\begin{aligned} &f^*_{1,y} = 0, && \text{for} \quad y = 0; \quad \infty \leqslant x \leqslant u_0 - \frac{\tau^2}{2} \\ &f^*_{1,y} = k\tau, && \text{for} \quad y = 0; \quad u_0 - \frac{\tau^2}{2} \leqslant x \leqslant u_0 - \frac{\tau^2}{2} + 1 \\ &f^*_{1,y} = 0, && \text{for} \quad y = 0; \quad 1 + x_0 - \frac{\tau^2}{2} \leqslant x \leqslant \infty \end{aligned}\right\} \qquad (9')$$

$$\left.\begin{aligned} &f^*_{1,x} = f^*_{1,y} = 0, && \text{for} \quad \tau = 0 \\ &f^*_{1,x} = f^*_{1,y} = 0, && \text{for} \quad x^2 + y^2 \to \infty \end{aligned}\right\} \qquad (9'')$$

and so on for the other f^*_n (for $n \geqslant 2$).

We shall limit ourselves here to determine the first two terms of the development (6).

3. WEDGE PROFILE DETERMINATION OF THE FIELD CORRESPONDING TO f^*_0

This problem is substantially the problem of Cole[3], studied with different approximations also by Trilling and Walker[4], by Guderley and Yoshihara[5] by Yoshihara alone[6], by Mackie and Pack[7], by Aslanow[8], and again by Mackie and Helliwell[9] and by Mackie alone[10]: what is important now to observe is that from these researches there appears a most noticeable insensibility of the pressure distribution on the wedge to the form of the sonic line, and to anything that may happen downstream thereof. Thus, for instance, while the original research by Cole admits that the sonic line is a straight line perpendicular to $\vec{q}_\infty$ through the shoulder B' of the wedge, that of Trilling and Walker determines the said line by fulfilling the conditions at the contour in the corresponding problem of Tricomi; and the results thus obtained are not correct only in that the shock wave is not determined, which however starts from B', and for $M_\infty < L$ influences also the subsonic portion of the field; now, the values of the drag coefficients obtained in both researches are very slightly different to one another: in fact, it is $\dfrac{(\gamma+1)^{1/3}}{\delta^{5/3}}\, c_r = 0{\cdot}3$ (Cole); $0{\cdot}275$ (Tr. and W.) for $\dfrac{1-M_\infty^2}{(\gamma+1)^{2/3}\delta^{2/3}} = 0{\cdot}825$. Further: for $M_\infty = 1$ Cole obtains $\dfrac{(\gamma+1)^{1/3}}{\delta^{5/3}}\, c_r = 1{\cdot}67$, whereas Gaderley and Yoshihara give the exact value (in the approximations of Tricomi), $(\gamma+1)^{1/3}\delta^{-5/3}\, c_r = 1{\cdot}75$ and Mackie and Pack evaluate the correct value (by assuming the equation of Chaplygin as the equation of motion) $(\gamma+1)^{1/3}\delta^{-5/3}c_r = 1{\cdot}55$. Still more significant is the comparison between the results of Cole and those of Mackie and Imai[11] for the problem of Helmholtz-Kirchoff relating to the same wedge, for which the sonic line is also the stream line to which the contour of the latter belongs, and which gets detached from the shoulder β': in fact, for $M_\infty = 1$, and against the value of c_r, according to Cole, already indicated above, we have $(\gamma+1)^{1/3}\delta^{-5/2}c_r = 1{\cdot}89$ according to Mackie, or 1·84 according to Imai. Finally, the values of the pressure coefficients on the wedge, for $M_\infty = 1$, obtained by Cole are in agreement, within 1 or 2 per cent, with those of the more exact research by Guderley and Yoshihara.

This insensibility of the pressure distribution to the shape of the sonic line and the configuration of the field downstream thereof, which appears most clearly from the set of results indicated above for the case of stationary motion, authorizes us to think that also for the instant research relating to unsteady motion any permissible assumption on said line will not have any material effect on the consequences deriving therefrom, as well as it appears permissible to assume that the form of the field adjacent to the obstacle can be determined independent of the shock wave and on anything which may happen in the supersonic portion.

For this reason we will assume, for the determination of f_0^*, the same assumption of Cole, whereas for f_1^* (and possibly for the other f_x^*) no particular condition will be imposed on the sonic line. Let us accordingly assume, as the independent variables, $u^* = -\dfrac{\partial f_0^*}{\partial x}$ and $v = \dfrac{\partial f_0^*}{\partial y}$ instead of x and y, and as the dependent variables the same x and y instead of f_0^*; Eq. (8) is thus transformed in the following:

$$u^* \frac{\partial^2 y}{\partial v^2} + v \frac{\partial^2 y}{\partial u^{*2}} = 0 \tag{10}$$

whilst it is

$$\frac{\partial x}{\partial v} = \frac{\partial y}{\partial u^*} \tag{10'}$$

Due to the conditions (8′), to the points $c_{-\infty}$, $c_{+\infty}$ there corresponds in the hodographic plane (u^*, v), the point c_∞ of abscissa u_p^*; in correspondence with 0′ the relative velocity is null, and therefore it will be

$$\frac{\partial f_0^*}{\partial x} = -(\gamma+1)^{1/3}\left(\frac{\delta}{M_\infty}\right)^{-2/3}\left[1+\frac{r}{\gamma+1}(1-M_\infty)+k\tau\right]$$

whence, by virtue of assumption (7)

$$\left(-\frac{\partial f_0^*}{\partial x}\right)_{0'} = (u^*)_{0'} = (\gamma+1)^{1/3}\left(\frac{\delta}{M_\infty}\right)^{-2/3}$$

which is close, with the usual condition in the theory of "small perturbations", to $u^* \to \infty$ in the top $0' \left(y = 0; \quad x = x_0 - \frac{\tau^2}{2} \right)$.

Correspondingly, the second of the conditions (8′) must be replaced by

$$f^*_{0,Y} = 1 + \frac{2}{\gamma+1}(1 - M_\infty) = v_0, \quad \text{for} \quad y = 0; \quad x_0 = \frac{\tau^2}{2} \leqslant x \leqslant 1 + x_0 - \frac{\tau^2}{2} \tag{11}$$

so that while to the line $c_{-\infty}0'$ in the physical plane there corresponds the line $c_\infty 0'_\infty$ of axis u^* in the hodographic plane, the segment $0'B'$ of the first-mentioned plane is homologous to the line $0'_\infty b'$ of equation $v = 1 + \frac{2}{\gamma+1}(1 - M_\infty) = v_0$ in the second plane, the point b' of the sonic line ($u^* = 0$) corresponding to the shoulder of the wedge. According to the assumption of Cole, the $u^* = 0$ is the straight line $x = x_0 + 1 - \frac{\tau^2}{2}$, and therefore it appears that to the domain of the physical plane upstream

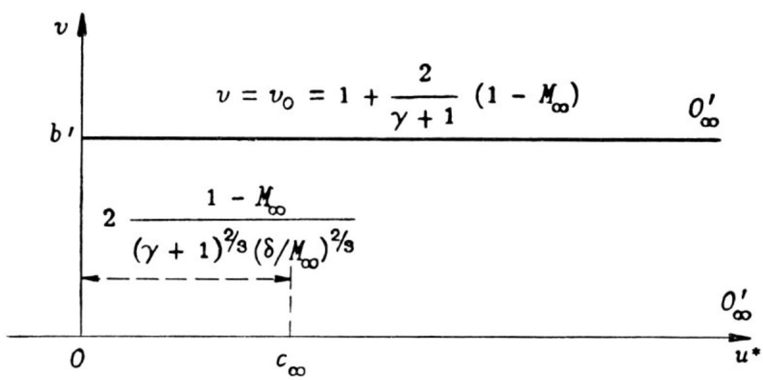

FIG. 2.

of the sonic line—wherein the f_0^* must be determined—there corresponds, in the hodographic plane, the indefinite half-strip indicated in Fig. 2. The y, which in this half-strip is defined by (10), fulfils—as indicated above—the equations:

$$\left.\begin{array}{lll} Y = 0 & \text{for} & 0 \leqslant u^* \leqslant \infty; \quad v = 0 \\ Y = 0 & \text{for} & 0 \leqslant u^* \leqslant \infty; \quad v = v_0 \\ Y_{u^*} = 0 & \text{for} & u^* = 0 \end{array}\right\} \tag{12}$$

whilst in correspondence with i_∞ it exhibits a singularity of the "doublet" type. The solution of (10), which fulfils all the conditions just mentioned, has been determined by Cole[3]; we have (Bryson[12]):

$$Y = v_0^{-1/3}\begin{cases} 2\left(\dfrac{2}{3}\dfrac{2}{v_0}\dfrac{2_\infty}{v_0}\right)^{1/3}\displaystyle\sum_{u=1}^{\infty} n\pi \sin\left(n\pi\frac{v}{v_0}\right) I_{-1/3}\left(n\pi\frac{2}{v_0}\right) K_{-1/3}\left(n\pi\frac{2_\infty}{v_0}\right) & z_\infty > z > 0 \\ 2\left(\dfrac{2}{3}\dfrac{2}{v_0}\dfrac{2_\infty}{v_0}\right)^{1/3}\displaystyle\sum_{u=1}^{\infty} n\pi \sin\left(n\pi\frac{v}{v_0}\right) K_{-1/3}\left(n\pi\frac{2}{v_0}\right) I_{-1/3}\left(n\pi\frac{2_\infty}{v_0}\right) & z > z_\infty > 0 \end{cases}$$

$$x = \begin{cases} l-\dfrac{\tau^2}{2}+2\left(\dfrac{z_\infty}{v_0}\right)^{1/3}\left(\dfrac{z}{v_0}\right)^{2/3}\displaystyle\sum_{u=1}^{\infty} n\pi \cos\left(n\pi\frac{v}{v_0}\right) I_{2/3}\left(n\pi\frac{z}{v_0}\right) K_{-2/3}\left(n\pi\frac{z_\infty}{v_0}\right) & z_\infty > z > 0 \\ -\dfrac{\tau^2}{2}-2\left(\dfrac{z_\infty}{v_0}\right)^{1/3}\left(\dfrac{z}{v_0}\right)^{2/3}\displaystyle\sum_{u=1}^{\infty} n\pi \cos\left(n\pi\frac{v}{v_0}\right) K_{2/3}\left(n\pi\frac{z}{v_0}\right) I_{-1/3} n\pi\frac{z_\infty}{v_0}) & z > z_\infty > 0 \end{cases} \tag{13}$$

4. WEDGE PROFILE—DETERMINATION OF f_1^*

Also for this determination we assume as the independent variables the same u^* and v defined in the preceding section, which are in biunivocal correspondence with x and y by virtue of (13). We obtain

$$L[f_1^*(u_1 v)] = \beta_{1,1} f^*_{1,uu} + 2\beta_{1,2} f^*_{1,uv} + \beta_{2,2} f^*_{1,vv} + \beta_1 f^*_{1,u} + \beta_2 f^*_{1,v}$$

wherein

$$\beta_{1,1} = -u\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{1}{T}$$

$$\beta_{1,2} = -u\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0$$

$$\beta_{2,2} = -u\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = -\frac{u}{T}$$

$$\beta_1 = -u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = 0$$

$$\beta_2 = -u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = 0$$

On the other hand, it is

$$f^*_{0,x\tau} = \frac{\partial u}{\partial \tau} = -\frac{\dfrac{\partial x}{\partial \tau}\dfrac{\partial y}{\partial v}}{T}$$

being, on account of (13) $\dfrac{\partial y}{\partial \tau} = 0$, while we have indicated with $T = \dfrac{\partial(x, y)}{\partial(u, v)}$ the Jacobian determinant of the transformation.

We obtain, therefore, from (9)

$$f^*_{1,uu} + u^* f^*_{1,vv} = \tau \frac{\partial y}{\partial v} \tag{14}$$

Concerning the conditions which f_1^* must fulfil on the frontier of the domain (u^*, v) wherein it is defined, we observe that it can be deduced that for $v = v_0$ it must be $K\tau = f^*_{z,v}\left(\dfrac{\partial v}{\partial y}\right)_{v=v_0}$, because $\dfrac{\partial v}{\partial x} = 0$ on $v = v_0$; we thus deduce:

$$f^*_{1,v} = K\tau(y_v)_{v=v_0} = K\tau g(u^*), \qquad \text{for} \qquad v = v_0 \tag{15}$$

wherein $g(u^*) = (y_v)_{v=v_0}$ can be calculated easily by means of the first of Eqs. (13). Similarly, we obtain:

$$f^*_{1,v} = 0, \qquad \text{for} \qquad v = 0 \tag{15'}$$

whilst no singularity exhibits the f_1^* for $v = 0$; $u^* = u^*_\infty$. A particular integral of the non-homogeneous equation is

$$f_1^{*(1)} = \tau \sum_{u=1}^{\infty} \cos\left(n\pi \frac{v}{v_0}\right) F_n(u^*) = \tau \psi^{(1)}(u^*, v) \tag{16}$$

wherein $F_n(u^*)$ is given by

$$F_n(u^*) = z v_0^{-4/3} \left(\frac{2}{3}\frac{z_\infty}{v_0}\right)^{1/3} I_{-1/3}\left(n\pi \frac{z_\infty}{v_0}\right) (n\pi)^2 \left(\frac{3}{2}\right)^{-1/3} \left(\frac{z}{v_0}\right)^{1/3} \times$$

$$\times \left[-K_{-1/3}\left(n\pi \frac{z}{v_0}\right) \int_{z_\infty}^{z} \zeta^{2/3} I_{-1/3}\left(n\frac{\pi}{v_0}\zeta\right) K_{-1/3}\left(n\pi\frac{\zeta}{v_0}\right) \mathrm{d}\zeta + \right.$$

$$\left. + I_{-1/3}\left(n\pi\frac{z}{v_0}\right)\left(\int_{z_\infty}^{z} \zeta^{2/3} K^2_{-1/3}\left(n\pi\frac{\zeta}{v_0}\right)\mathrm{d}\zeta - \int_{z_\infty}^{\infty} \zeta^{2/3} K^2_{-1/3}\left(n\pi\frac{\zeta}{v_0}\right)\mathrm{d}\zeta\right)\right] \tag{17}$$

for $z > z_\infty$; or

$$F_n(u^*) = z v_0^{-4/3}\left(\frac{2}{3}\frac{z_\infty}{v_0}\right)^{1/3} K_{1/3}\left(n\pi\frac{z_\infty}{v_0}\right)(n\pi)^2\left(\frac{3}{2}\right)^{-1/3}\left(\frac{z}{v_0}\right)^{1/3}\times$$

$$\times\Bigg\{-K_{-1/3}\left(n\pi\frac{z}{v_0}\right)\int_{z_\infty}^{z}\xi^{2/3}I^2_{-1/3}\left(n\frac{\pi}{v_0}\xi\right)\mathrm{d}\zeta$$

$$+I_{-1/3}\left(n\pi\frac{z}{v_0}\right)\int_{z_\infty}^{z}\zeta^{2/3}K_{-1/3}\left(\frac{n\pi}{v_0}\zeta\right)I_{-1/3}\left(\frac{n\pi}{v_0}\zeta\right)\mathrm{d}\zeta$$

$$-I_{-1/3}\left(un\frac{z}{v_0}\right)\frac{\int_{z_\infty}^{z}\zeta^{2/3}K^2_{-1/3}\left(\frac{un}{v_0}\zeta\right)\mathrm{d}\zeta}{K_{-1/3}\left(\frac{un}{v_0}z_\infty\right)}I_{-1/3}\left(un\frac{z_\infty}{v_0}\right)\Bigg\} \tag{17'}$$

for $z < z_\infty$.

The series which expresses (16) will absolutely converge both for $u^* < u^*_\infty$ and for $u^* > u^*_\infty$, and the series corresponding to the first interval is the analytic extension of that corresponding to the second interval. On the other hand, $f^{*(1)}_{1,v}$ fulfils (15′), whereas for $v = v_0$ it is $f^{*(1)}_{1,v} = 0$. It is therefore necessary to add to (16) a solution of the homogeneous equation

$$f^*_{1,u^*u^*} + v f^*_{1,vv} = 0$$

which fulfils (15) and for which it is also $(f^*_{1,v})_{v=0} = 0$.

To this end we assume

$$f_1^{*(2)} = K\tau y^{(2)}(u^*, v) = K\tau\int_0^\infty G(\lambda)\frac{\cosh \mathrm{d}u}{\sinh \mathrm{d}v_0}S(u^*, \lambda)\,\mathrm{d}\lambda \tag{18}$$

with

$$S(o, \lambda) = 0; \quad \left(\frac{\partial S}{\partial u^*}\right)_{u^*=0} = 1 \tag{18'}$$

and $G(\lambda)$ an arbitrary function, momentarily subject to the sole condition that $G(o) = 0$ (see for instance Mackie[10]). The S fulfils

$$\frac{\mathrm{d}^2 S}{\mathrm{d}u^{*2}} + \lambda^2_{u^{*2}} S = 0 \tag{18''}$$

and therefore, due to the conditions (18′), it will be

$$S(u^*, \lambda) = z^{2/3}\Gamma\left(\frac{5}{3}\right)\left(\frac{1}{3}\lambda\right)^{-2/3}\left(\lambda\frac{2}{3}u^{*3/2}\right)^{1/3} T_{1/3}\left(\frac{2}{3}\lambda u^{*3/2}\right) \tag{19}$$

From (18) it appears that $f^{*(2)}_{1,0} = 0$, for $v = 0$: by imposing that for $v = v_0$ the condition (15) must be fulfilled, we obtain

$$\int_0^\infty \lambda G(\lambda)\, S(u^*, \lambda)\, \mathrm{d}\lambda = g(u^*) \tag{20}$$

The above (20) is an integral equation for $G(\lambda)$, which for $S(u^*, \lambda)$, as defined by (18″) and (18′), has been solved by Germain[13], and in particular we have

$$G(\lambda) = z\left(\frac{1}{3}\right)^{5/3}\left[\Gamma\left(\frac{4}{3}\right)\right]^{-2}\lambda^{2/3}\int_0^\infty g(u^*)\, S(u^*, \lambda)\, \mathrm{d}u^* \tag{21}$$

For the expression of $g(u^*)$, as it can be deduced from (13), the integral in the R.H. member of (21) will converge, and it appears further that it is $G(o) = 0$, as required. If therefore we assume

$$f_1^* = f_1^{*(1)} + f_1^{*(2)}$$

f_1^* will fulfil all the conditions imposed therefore.

5. WEDGE PROFILE. PRESSURE COEFFICIENTS ON THE PROFILE. COMPARISON WITH THE RESULTS OF THE LINEAR THEORY

In the approximation corresponding to (1), the pressure coefficient in a generical point of the profile is given by

$$C_p = -z(\gamma+1)^{-1/3}\delta^{2/3} f_x \tag{22}$$

and therefore, on account of (7),

$$C_p = z(\gamma+1)^{-1/3}\delta^{2/3} f^*_{0,x}[1 + K^*\tau\Psi^{(1)}_u(u, v_0) + K\Psi^{(2)}_n(u, v_0)] - \frac{2}{\gamma+1}(1 - M^2_\infty) \tag{22'}$$

If now we calculate the values of C_p for the instant in correspondence of which the velocity relative to the profile is equal to the critical velocity a^*, corresponding to the stream at the infinite, we obtain for this instant

$$K\tau_0 = \frac{2}{\gamma+1}(1 - M_\infty)$$

and therefore

$$K^*\tau_0 = \frac{2}{(\gamma+1)^{5/3}}\frac{1}{D}$$

which is independent of the parameter of unsteadiness, and it will be, for $\tau = \tau_0$,

$$(C_p)_{\tau=\tau_0} = -2(\gamma+1)^{-1/3}\delta^{2/3} f^*_{0,x}\left[1 + \frac{2}{(\gamma+1)^{5/3}}\frac{1}{D}\Psi^{(1)}_{u^*}(u^*, v_0) + \right.$$
$$\left. + \frac{2}{(\gamma+1)^{5/3}}\frac{1}{D}K\Psi^{(2)}_{u^*}(u, v_0)\right] - \frac{2}{\gamma+1}(1 - M^2_\infty) \tag{23}$$

which appears to cause the pressure coefficient—for an asymptotic relative velocity equal to the critical velocity—to depend linearly on the parameter K.

It is interesting to compare the result obtained with that given by the linear theory: now, from the research of Gardner and Ludloff(14) (see also Ch. Roumien(15)) we have

$$C_p = \delta K^{-1/2} H_0(x)[1+KH_1(x)+-] \qquad (23')$$

what, unlike (23), for $k \to 0$ gives $c_r \to \infty$ in the whole profile. If however, in accordance with (2), we assume (24) $K = D_1 S^{2/3}$, it will be $\delta K^{-1/2} \approx \delta^{2/3}$ and (23') becomes (23'')

$$C_0 = \text{const.}\ \delta^{2/3} H_0(x)[1+H_1(x)K+-] \qquad (23'')$$

which is formally analogous to (23); the analogy of behaviour of (23'') to (23) also exists, at least qualitatively, as far as the dependence of C_p on x is concerned, since in both cases it is $C_p \to \infty$ as the point on the profile tends to the apex 0' of the latter. The difference, from the quantitative point of view, in this connection consists in that in (23''), (25) $H_0(x) \approx x^{-1/4}$ whereas $f^*_{0,x}$, indicating for an instant again with x the distance of the point on the wedge from 0', also tending to infinite when x tends to zero, however less rapidly than expressed by (25); we may say, $f^*_{0,x} \approx x^{-1/n}$ with $n > 4$.

The research of Gardner and Ludloff is carried out for the case of a profile decelerating until the sonic velocity is reached, starting from supersonic initial conditions: but Biot(16), who has considered the case of a profile which accelerates from zero up to the velocity of sound, obtained—as far as the dependence of the pressure coefficient on K and x is concerned—results indentical to those indicated above.

We may still observe that the passage from (23') to (23'') by virtue of (24), is wholly analogous to that which, by utilizing the parameter of transonic similitude, permits the passage from the the expression which gives C_p, in the stationary motion, according to the rule of Prandtl-Glauert, to that which gives the same according the similitude rule of von Kármán. The parameter $D_1 = \dfrac{K}{\delta^{2/3}}$, or, what is the same, $D^* = \dfrac{K}{1-M_\infty} = D_1 D$; it appears therefore to have the meaning of a non-stationary parameter of transonic similitude.

6. INFLUENCE OF UNSTEADINESS ON THE SHAPE AND FORMATION OF THE SHOCK WAVE (FOR THE TRANSLATORY MOTION OF THE PROFILE).

In the preceding example the shock wave is always anchored to the shoulder of the wedge and exerts a very slight influnce on the pressure distribution on the contour thereof, upstream of β': just this property has permitted the obtain of a solution, although approximate, of the problem. However, in more general cases of motion, the shock wave or waves a most considerable importance for the determination of the aerodynamic characteristics of the profile, and it is therefore essential to determine how they are influenced by the non-stationarity. In the case of the latter it is due to rotational oscillations about a center lying on the straight line to which the chord of the profile belongs, the problem has been studied in two most interesting works written by Messrs. Coupry and Piazzoli[17] and Eckhauss[18]: unfortunately, this study exhibits much higher difficulties in the case here considered, wherein the stationarity is due to the accelerated translatory motion of the obstacle in the direction of $\vec{q}_\infty$, and therefore we are constrained to consider only some particular aspects of the problem.

One of these aspects relates to the shape of the shock wave and the formation thereof in accelerated motion, and in order to deduce some property of the flow we shall try to determine local solutions of the equation of motion which are valid directly upstream and directly downstream of the shock wave.

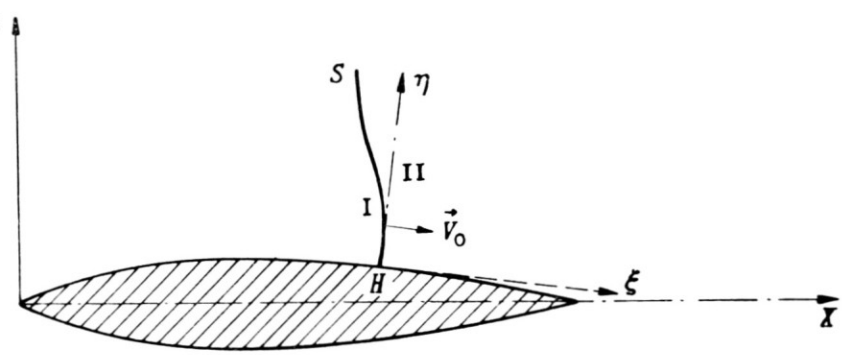

FIG. 3.

If H is the point of the profile from which, in a generical instant, the shock wave starts (Fig. 3), and (x_h, y_h) are its coordinates, we consider a system of axes (ξ, η) having its origin, in any instant, coincident with H (ξ tangent to the profile in H, whilst q is normal to ξ); thus, we have

$$x = x_h(\tau)+\xi; \quad y = y_h(\tau)-\xi\theta_h+\eta \tag{26}$$

θ_h being the slope of the tangent in H on the x axis. Let us now assume:

$$\overline{\psi}(x, y, \tau) = \overline{\phi}\,[x_h(\tau)+\xi, y_h(\tau)-\xi_h\theta+\eta] = \overline{\overline{\phi}}(\zeta, \eta, \tau) \tag{27}$$

We deduce

$$\overline{\phi}_x \cong \overline{\overline{\phi}}_\xi; \quad \overline{\phi}_y = \overline{\overline{\phi}}_y; \quad \overline{\phi}_\tau = \overline{\overline{\phi}}_\tau - \frac{\mathrm{d}x_h}{\mathrm{d}\tau}\overline{\overline{\phi}}_\xi$$

$$\overline{\phi}_{\tau,x} = \overline{\overline{\phi}}_{\tau,\xi} - \frac{v_0}{q_\infty K}\overline{\overline{\phi}}_{\xi\xi}$$

being v_0 the velocity of displacement of H, namely of the wave front on the profile. The equation to which $\overline{\overline{\phi}}$ fulfils is

$$2K\overline{\overline{\phi}}_{\tau\xi} + \left[(\gamma+1)\,\overline{\overline{\phi}}_\xi - \frac{2v_0}{q_\infty}\right]\overline{\overline{\phi}}_{\xi\xi} - \overline{\overline{\phi}}_{\eta\eta} = 0 \tag{28}$$

Let

$$\xi = \xi(s,\tau); \quad \eta = \eta(s,\tau) \tag{29}$$

be the parametric equations of the line S which is the wave front, s being the length of the arc measured S starting from H; if we now indicate with u and v the velocity components of the motion relative to the profile along the axes (ζ, η), and characterized with the indexes 1 and 2 the values of the magnitudes directly before and after the shock, we obtain, after not difficult but rather toilsome calculations:

$$\left(\frac{\partial}{\partial\eta}\frac{u_2}{a_r^*}\right)_H + \left[\frac{a_r^{*2}}{u_r^2}\frac{\partial}{\partial\eta}\left(\frac{u_1}{a_r^*}\right)\right]_H = \left(\frac{\mathrm{d}^2\eta}{\mathrm{d}s^2}\right)_H A \tag{30}$$

wherein A is given by

$$A = \frac{u_1}{a_r^*}\left\{2\frac{\gamma-1}{\gamma+1} - \frac{a_r^{*2}}{u_1^2} - \frac{1+2\left(1-M_2^2+\dfrac{2v_0}{q_\infty}\right) - \dfrac{a_r^{*2}}{u_2^2}\left(2-M_2^2+\dfrac{2v_0}{q_\infty}\right)}{1-M_2^2+\dfrac{2v_0}{q_\infty}}\right\} \tag{31}$$

being

$$1-M_2^2 \cong (\gamma+1)\left(1-\frac{a_r^*}{u_1}-\frac{2v_0}{q_\infty}\right) \tag{31'}$$

Eq. (30) is quite similar to that obtained for the corresponding stationary problem, and actually it can be reduced to the latter by assuming $v_0 = 0$. The consequences deriving from this assumption are therefore also identical to those already known from the permanent motion; when H is approached on the profile, it is certainly

$$\left[\frac{\partial}{\partial\eta}\left(\frac{u_2}{a_r^*}\right)\right]_H = \left(\frac{u_2}{a_r^*}\frac{1}{R_c}\right)_H \cong \left[\frac{a_r^*}{u_1}\left(1+\frac{2v_0}{q_\infty}\right)\frac{1}{R_c}\right]_H$$

$(R_c)_H$ being the curvature radius of the profile in H (divided by l); likewise, it is

$$\left[\frac{\partial}{\partial\eta}\left(\frac{u_1}{a_r^*}\right)\right]_H = \left(\frac{u_1}{a_r^*}\right)_H \left(\frac{1}{R_c}\right)_H$$

and accordingly (30) becomes

$$\left[2\frac{a_r^*}{u_1}\left(1+\frac{v_0}{q_\infty}\right)\frac{1}{R_c}\right]_H = A\left(\frac{\mathrm{d}^2\eta}{\mathrm{d}s^2}\right)_H \tag{30'}$$

Now, if the profile has its concavity towards the stream, it is $(R_c)_H > 0$, and being $A < 0$ always for $\frac{u_1}{a^*} > 1$ for $s = 0$ it will be $\left(\frac{\mathrm{d}^2\eta}{\mathrm{d}s^2}\right)_H < 0$, as it must actually be, whereby the shock line in H can be effectively determined so that the condition (30′) will be fulfilled. If instead the profile is convex, and therefore $(R_c)_H < 0$, $\left(\frac{\mathrm{d}^2\eta}{\mathrm{d}s^2}\right)_H$ by virtue of (30′) should be positive, and because $\left(\frac{\mathrm{d}\eta}{\mathrm{d}s}\right)_H = 1$ whereas it must be everywhere $\frac{\mathrm{d}\eta}{\mathrm{d}s} \leqslant 1$, it appears that there is no normal shock wave, in contact with the profile, comparible with the condition (30′). Since it is certainly true that

$$\left[\frac{\partial}{\partial\eta}\left(\frac{u_1}{a_r^*}\right)\right]_H = \left(\frac{u_1}{a_r^*}\frac{1}{R_c}\right)_H$$

it results that the value of $\left(\frac{\partial}{\partial\eta}\frac{u_1}{a_r^*}\right)_H$, when H is approached on S, is given, on account of (30), by

$$\left[\frac{\partial}{\partial\eta}\left(\frac{u_1}{a_r^*}\right)\right]_H = -\left(\frac{a_r^*}{u_1}\frac{1}{R_c}\right)_H + A\left(\frac{\mathrm{d}^2\eta}{\mathrm{d}s^2}\right)_0 \tag{32}$$

whereas, if H is approached on the profile, it will be

$$\left[\frac{\partial}{\partial\eta}\left(\frac{u_r}{a_r^*}\right)\right]_H = \left(\frac{a_r^*}{u_1}\frac{1}{R_c}\right)_H \tag{32'}$$

and therefore $\frac{\partial}{\partial\eta}\left(\frac{u_r}{a_r^*}\right)$ should have in H a point of indetermination tending in H to different limits, depending on the line on which H is approached. Now, it is possible to obtain a local solution which will fulfil the condition just indicated, and it is also likely that, exactly because only one local solution is considered, there can exist more than one form of solution having the properties required; on the other hand, we think that also in the case when (30′) can be fulfilled, due to the singularities that the functions u_r and v_r would have in H, the shock line S must be determined not on account of (30′) but by seeking solutions in the two fields separated by S which would join—together with their normal derivatives—on the same, the shock equation being further verified on the latter. The local solutions, on both sides of the wave, which we propose here, are similar to those already indicated by us for the case of the stationary flow, and therefore we shall only give a brief hint.

Upstream of S, $\bar{\bar{\phi}}$ is regular and may be expressed by

$$\bar{\bar{\phi}} = \bar{\bar{\phi}}_H + \varepsilon\xi + \tfrac{1}{2}b_{1,1}\xi^2 + b_{1,2}\xi\eta + \tfrac{1}{2}b_{2,2}\eta^2 + \cdots \tag{33}$$

being

$$\varepsilon = \left(\frac{u_1 - a_r^*}{q_\infty}\right)_H; \quad b_{1,1} = \left[\frac{\partial}{\partial\xi}\left(\frac{u}{q_\infty}\right)\right]_H = (\bar{\bar{\phi}}_{\xi\xi})_H$$

$$b_{1,2} = (\bar{\bar{\phi}}_{\xi\eta})_H = (1+\varepsilon+\varepsilon_\infty)\left(\frac{1}{R_c}\right)_H$$

$$b_{2,2} = (\bar{\bar{\phi}}_{\zeta\eta})_H = 2K\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} + \left[(\gamma+1)\varepsilon - \frac{2v_0}{q_\infty}\right]b_{1,1} \tag{34}$$

if $q_\infty = a_r^*(1-\varepsilon_\infty)$.

Downstream of S we write Eq. (28) as follows:

$$\beta_0 + \varepsilon\beta_{2,2}\bar{\bar{\phi}}_{\xi\xi} + \bar{\bar{\phi}}_{\eta\eta} = 0 \tag{35}$$

wherein

$$\beta_0 = 2K(\bar{\bar{\phi}}\tau\,\xi)_H = 2K\frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\varepsilon + \frac{2v_0}{q_\infty}\right)$$

$$\beta_{2,2} = -\frac{1}{\varepsilon}\left[(\gamma+1)\bar{\bar{\phi}}_\xi - \frac{2v_0}{q_\infty}\right]_H = (\gamma+1)\left[1 - \frac{2\gamma}{\gamma+1}\,\frac{v_0}{\varepsilon q_\infty}\right]$$

Upon assuming, in immediate proximity of H,

$$\bar{\bar{\phi}}(\xi,\eta) = \frac{\beta_0}{2}(\xi^2+\eta^2) + \phi(\bar{\xi},\eta); \tag{36}$$

$$\bar{\xi} = \frac{\xi}{\sqrt{\varepsilon}\,\sqrt{\beta_{2,2}}}$$

Eq. (35) is transformed into

$$\phi_{\bar{\xi}\bar{\xi}} + \phi_{\eta\eta} = 0$$

Now, we assume that

$$\left.\begin{aligned} \phi &= R,\, W(\bar{\xi}+i\eta) = R,\, W(\zeta) \\ W(2) &= w_0,\, \zeta+\lambda,\, \zeta^2\log\zeta + i\lambda_2\zeta^2 + P(\zeta) \end{aligned}\right\} \tag{37}$$

wherein $P(\zeta)$ is a regular function for $\zeta = 0$, namely in H, which fulfils the conditions $\left(\frac{\mathrm{d}P}{\mathrm{d}\zeta}\right)_{\zeta=0} = \left(\frac{\mathrm{d}^2P}{\mathrm{d}\zeta^2}\right)_{\zeta=0} = 0$, and which does not require to be determined in order to obtain the local solution, to which we confine ourselves here; λ_1 and λ_2 are two constants to be determined so as to fulfil the conditions imposed on the flow in H, namely to the (32) and (32′): finally, it is

$$w_\infty = \varepsilon^{3/2}\sqrt{\beta_{2,2}}\left(-1+\frac{2v_0}{\varepsilon q_\infty}\right)$$

By equalling the two expressions of $\bar{\bar{\phi}}$ given, by (33) and (36); we obtain the equation of S, while the junction of the normal derivatives on S is ensured, in immediate proximity of H, by the condition imposed by (32).

The procedure is quite identical, as already stated above, to that indicated in (19) for the stationary case, and we again obtain that the shock line exhibits in H a logarithmic singularity, whilst the flow undergoes in H, immediately after the shock, a rapid expansion, since it is $\left(\frac{\partial u_2}{\partial \xi}\right)_H \equiv (\log \xi)$ in proximity of H. What is still of interest to note, as far as the influence on nonstationarity is concerned, is that ε, which is a measure of the excess of velocity on the critical velocity immediately upstream of the shock, is now multiplied by the term $1-\frac{2\gamma}{\gamma+1}\frac{V_0}{\varepsilon q_\infty}$, whereas the difference of the velocities u_1-u_2 is given by $u_1-u_2 \cong a_r^* 2\varepsilon\left(1-\frac{V_0}{\varepsilon q_\infty}\right)$. It appears therefore that what characterized such influence is the parameter $\frac{V_0}{\varepsilon q_\infty}$, and accordingly this influence is sensible if

$$\frac{V_0}{q_\infty} \equiv 0(\varepsilon) \tag{38}$$

Now, in nearly stationary conditions, it is

$$\frac{V_0}{q_\infty} = \frac{1}{q_\infty}\frac{\mathrm{d}X_a^*}{\mathrm{d}t^*} = \frac{2}{\gamma+1}M_\infty\frac{\mathrm{d}Xa}{\mathrm{d}M_\infty}K^2$$

and since it is $M_\infty\frac{\mathrm{d}Xa}{\mathrm{d}M_\infty} \equiv 0(1)$, it will be

$$\frac{V_0}{q_\infty} \equiv 0(K^2) \tag{38′}$$

and the condition indicated previously gives

$$K \equiv 0(\varepsilon^{1/2}) \tag{39}$$

Eq. (39) is obviously valid only for K so small that the flow may be considered as nearly stationary; let us now examine the reverse limit-case of $K \to \infty$ corresponding to a sudden variation of the velocity of the stream relative to the profile. Let us schematize it as a flat wall with an angle of attack α (Fig. 4), and let Δq_∞ the velocity suddenly assumed by this wall which has remained fixed up to the instant $t^* = 0$. An abrupt increment of the component of the relative velocity normal to the obstacle

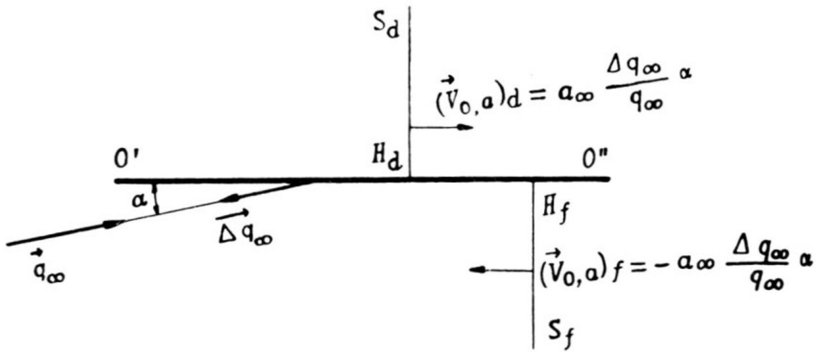

FIG. 4.

$\Delta q_u = -aq_\infty \alpha$ is thus produced, to it there corresponds a sudden increment of the density of distribution of the bound vorticity, which according to the result obtained by Possio[20] becomes

$$\Delta\Gamma = 2a_\infty \frac{\Delta q_\infty}{q_\infty} \alpha \tag{40}$$

We have thus an increase of the velocity on the face $\Delta u_f = -a_\infty \dfrac{\Delta q_\infty}{q_\infty} \alpha$, and an increase of velocity on the back $\Delta u_d = a_\infty \dfrac{\Delta q_\infty}{q_\infty} \alpha$. If now on the back, in correspondence with a point H, in the original flow a shock wave is present, for $t = 0^*$ the two velocities before and after the shock are bound by $u_1 u_2 \leqslant a^{*2}$, whereas for $t = 0^*$, and always with reference to the same system of axes, we have $\left(u_1 + a_\infty \dfrac{\Delta l_\infty}{q_\infty} \alpha - V_{0,a}\right)\left(u_2 + a_\infty \dfrac{\Delta q_\infty}{q_\infty} \alpha - V_{0,a}\right) = a^{*2}$, $V_{0,a}$ being the propagation velocity speed of the wave with respect to the fixed system of axes. We deduce (41) $V_{0,a} = a_\infty \dfrac{\Delta q_\infty}{q_\infty} \alpha$; consequently, the propagation velocity relative to the profile is: $V_0 = \Delta q_\infty \left(\dfrac{1\alpha}{M_\infty} + 1\right)$; if the wave starts from the front, it is instead

$$V_{0,a} = \Delta q_\infty \left(-1 \frac{1}{M_\infty} \alpha\right) \tag{41''}$$

This difference between the propagation velocities of the back and front waves may indicate a beneficial effect of the acceleration of the critical transonic condition, for the profile with incidence. Since it is $\Delta q_\infty = q_\infty \lim_{\substack{K\to\infty \\ \tau\to 0}} (K\tau)$ the (41) seem to indicate a dependence of V_0 on τ, and that $V_0/q_\infty \equiv 0(K\tau)$ for values of K.

7. APPROXIMATE DETERMINATION OF NON-STATIONARY TRANSONIC FLOWS

The extreme complication involved in the problem of the stationary flows, for the values of K corresponding to (2) makes useful the research for procedures and methods based on simplifications, which might appear as rough approximation, but from which have been obtained very satisfactory results for the corresponding stationary problem. To this purpose, particularly useful is the method of Spreiter and Alksne[(1)], at least for values of $M\infty$ sufficiently close to one, provided that the shock waves are either on the trailing edge or very close to this point.

From (1), by omitting the term in K^2 because for the procedure just mentioned only the flow in close proximity of the profile is considered, and if

$$x = X-\tfrac{1}{2}\tau^2; \quad f(x,y,\tau) = f^{**}(x,y,\tau) \tag{42}$$

we deduce

$$\frac{2K}{(\gamma+1)^{2/3}\delta^{1/3}} f^{**}_{x,\tau} + \left[\frac{M_\infty^2-1+2K\tau}{(\gamma+1)^{2/3}\delta^{2/3}} + f^{**}_x\right] f^{**}_{xx} - f^{**}_{yy} = 0 \tag{43}$$

Assuming $f^{*}_{xx} = \lambda$; and, if the equation of the profile contour is $y = \delta g(x)$, so that for the tangency of the velocity relative to the profile of said contour it must be $(f^{**}_y)_{ys_1} = g_x(1+K\tau)$ for $0 \leqslant X \leqslant 1$, (43′) let us now re-write Eq. (43) in the following form:

$$-\frac{2K}{(\gamma+1)^{2/3}\delta^{2/3}} f^{**}_{x,\tau} - \lambda f^{**}_x + f^{*}_{yy} = \left[-\frac{1-M_\infty^2-2K\tau}{(\gamma+1)^{2/3}\delta^{2/3}}\right]\lambda = F \tag{43″}$$

The simplification used by Spreiter and Alksne is, as it is well known, to admit λ—in the equation corresponding to (43″) for the stationary flow—to be constant, i.e. independent of x and y; let us now extend this simplification also with respect to τ. Then, by applying the Laplace transformation with respect to the variable τ, we obtain

$$\bar{f}^{**}_{yy} - \lambda \bar{f}^{**}_x - \frac{2K}{(\gamma+1)^{2/3}\delta^{2/3}}[p\bar{f}^{**}_x - (f^{**}_x)_0] = \bar{F} \tag{44}$$

being

$$\bar{f}^{**} = L_\tau[f^{**}]; \quad \bar{F} = L_\tau[F]; \quad (f^{**}_x)_0 = (f^{**}_x)_{\tau=0}$$

whilst from (43′) we deduce

$$\bar{f}_y^* = \frac{g_x}{p} + \frac{K}{p^2} g_x. \tag{45}$$

From (43″) we obtain, if

$$\bar{u} = (\bar{f}_x^{**})_{y=0}; \quad h = \frac{2K}{(\gamma+1)^{2/3}\delta^{2/3}} \tag{46}$$

$$\bar{u} = -\frac{1}{\sqrt{\pi(\lambda+hp)}}\left(\frac{1}{p}+\frac{u}{p^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{g\varepsilon}{\sqrt{x-\xi}}\,\mathrm{d}\xi -$$

$$-\frac{2}{\lambda+hp}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\mathrm{d}\eta\int_{-\infty}^{x}\sigma\bar{F}\,\mathrm{d}\xi + \frac{h}{\lambda+hp}\frac{\partial}{\partial x}\int_0^{\infty}\mathrm{d}\eta\int_{-\infty}^{x}\sigma(f_x^{**})_0\,\mathrm{d}\xi \tag{47}$$

wherein

$$\sigma = \sqrt{\frac{\lambda+hp}{u\pi(x-\xi)}}\, e^{-\frac{(\lambda+hp)\eta^2}{4(x-\xi)}}$$

After some calculation we have

$$\bar{u} = -\frac{1}{\sqrt{\pi(\lambda+hp)}}\left(\frac{1}{p}+\frac{K}{p^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{g_\xi\,\mathrm{d}\xi}{\sqrt{x-\xi}} + \frac{\lambda}{p(\lambda+hp)}\frac{1-M_\infty^2}{(\gamma+1)^{2/3}\delta^{2/3}} -$$

$$-\frac{2K\lambda}{(\gamma+1)^{2/3}\delta^{2/3}p^2}\frac{1}{\lambda+hp} + \frac{h}{\lambda+hp}(u)_0 \tag{48}$$

By anti-transforming (48) we obtain

$$u = -\frac{1}{\sqrt{M\lambda}}\left[1+K\tau-\operatorname{erfc}\left(\sqrt{\frac{\lambda\tau}{h}}\right)-K\int_0^\tau \operatorname{erfc}\sqrt{\frac{\lambda\tau^1}{h}}\,\mathrm{d}r^1\right]\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{g_\xi\,\mathrm{d}\xi}{\sqrt{x-\xi}}$$

$$+\frac{1-M_\infty^2-2K\tau}{(\gamma+1)^{2/3}\delta^{2/3}} - \frac{1-M_\infty^2}{(\gamma+1)^{2/3}\delta^{2/3}}e^{-\frac{\lambda\tau}{h}} + \frac{2Kh}{\lambda(\gamma+1)^{2/3}\delta^{2/3}}\left(1-e^{-\frac{\lambda\tau}{h}}\right) + e^{-\frac{\lambda\tau}{h}}(u) \tag{49}$$

Now, we may re-write Eq. (49) in the form

$$\left[u-\frac{1-M_\infty^2-2K\tau}{(\gamma+1)^{2/3}\delta^{2/3}}+\frac{1-M_\infty^2}{(\gamma+1)^{2/3}\delta^{2/3}}e^{-\frac{\lambda\tau}{h}}-(u)_0 e^{-\frac{\lambda\tau}{h}}\right]\left(\frac{\partial u}{\partial x}\right)^{1/2} =$$

$$= -\frac{1}{\sqrt{\pi}}\left[1+K\tau-\operatorname{erfc}\left(\sqrt{\frac{\lambda\tau}{h}}\right) = K\int_0^\tau \operatorname{erfc}\left(\sqrt{\frac{\lambda\tau^1}{h}}\right)\mathrm{d}\tau^1\right]\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{g_\xi\,\mathrm{d}\xi}{\sqrt{x-\xi}}$$

$$+\frac{2Kh(1-e^{-\frac{\lambda\tau}{h}}}{\lambda^{1/2}(\gamma+1)^{2/3}\delta^{2/3}} \tag{49'}$$

obtained from (49′) by re-writing $\dfrac{\partial u}{\partial x}$ in place of λ, whereas in the terms which are small if $\dfrac{\lambda\tau}{h}$ is sufficiently great, the symbol λ has been maintained. Note now that for $\tau = 0$ we have

$$\left[(u)_0 - \frac{1-M_\infty^2}{(\gamma+1)^{2/3}\delta}\right]\left(\frac{\mathrm{d}(u)_0}{\mathrm{d}x}\right)^{M_2} = -\frac{1}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\int\limits_0^x \frac{g_\xi}{\sqrt{x-\xi}}\,\mathrm{d}\xi \tag{50}$$

whilst in nearly-stationary conditions ($K \ll 1$) it is

$$\left[u^{(s)} - \frac{1-M_\infty^2-2h\tau}{(\gamma+1)^{2/3}\delta^{2/3}}\right]\left(\frac{\partial u^{(5)}}{\partial x}\right)^{1/2} = -\frac{1}{\sqrt{\pi}}(1+K\tau)\frac{\mathrm{d}}{\mathrm{d}x}\int\limits_0^x \frac{g_\xi\, d_\xi}{\sqrt{x-\xi}} \tag{51}$$

Since the Mach number of the flow relative to the profile becomes equal to one for (51) $2K\tau = 1-M_\infty^2$, it appears that for the values of $K\tau$ of the order of magnitude as indicated by (51), $\left(\dfrac{\partial u^{(2)}}{\partial x}\right)$ and $\left(\dfrac{\mathrm{d}(u_0)}{\mathrm{d}x}\right)$ have values close enough to one another so that we may put in (48′)

$$\left[\frac{1-M_\infty^2}{(\gamma+1)^{2/3}\delta^{2/3}} - (u)_0\right] e^{-\frac{\lambda\tau}{h}}\left(\frac{\partial u}{\partial x}\right)^{1/2} = \frac{e^{-\frac{\lambda\tau}{h}}}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\int\limits_0^x \frac{g_\xi\,\mathrm{d}\xi}{\sqrt{x-\xi}}$$

and in lieu of λ the value $\lambda_5 = \dfrac{\partial u^{(1)}}{\partial x}$. Thus, we obtain

$$\left[u - \frac{1-M_\infty^2-2K\tau}{(\gamma+1)^{2/3}\delta^{2/3}}\right]^3 = 3\int\limits_{x_0}^x F^2(x'\ \tau)\,\mathrm{d}X' \tag{52}$$

being now

$$F(x,\tau) = \frac{1}{\sqrt{\pi}}\left[1+K\tau+\left(e^{-\frac{\lambda s\tau}{h}} - \operatorname{erfc}\sqrt{\frac{\lambda s\tau}{h}}\right) - \right.$$
$$\left. -K\int\limits_0^\tau \operatorname{erfc}\sqrt{\frac{\lambda_s\tau'}{h}}\,\mathrm{d}\tau'\right]\frac{\mathrm{d}}{\mathrm{d}x}\int\limits_0^x \frac{g_\xi}{\sqrt{x-\xi}} + \frac{2Kh(1-e^{-\frac{\lambda_s\tau}{h}})}{\lambda_s^{2/2}(\gamma+1)^{2/3}\delta^{2/3}} \tag{53}$$

while X_0 is the value of X_0 for which $F(X_0,\tau) = 0$

Correspondingly, we obtain for the pressure coefficient

$$c_p = 2(\gamma+1)^{-1/3}\delta^{2/3}\left\{\frac{1-M_\infty^2-2K\tau}{(\gamma+1)^{2/3}\delta^{2/3}} + \left[3\int\limits_{x_0}^x F^2(x^1,\tau)\,\mathrm{d}x^1\right]^{1/3}\right\} \tag{54}$$

Eq. (54) has been applied to calculate the pressure distribution on a profile having the form of an arc of a circle, at the time $\tau = \dfrac{1-M_\infty}{K}$, assuming

$$M_\infty = 0{\cdot}9;\quad \delta = 0{\cdot}05;\quad g(x) = 2x(1-x);\quad K = \sqrt{2}\cdot 10^{-2};\ 8\cdot 10^{-2};\ \sqrt{2}\cdot 10^{-1}$$

and the results are given by the diagrams of Figs. 5 and 6.

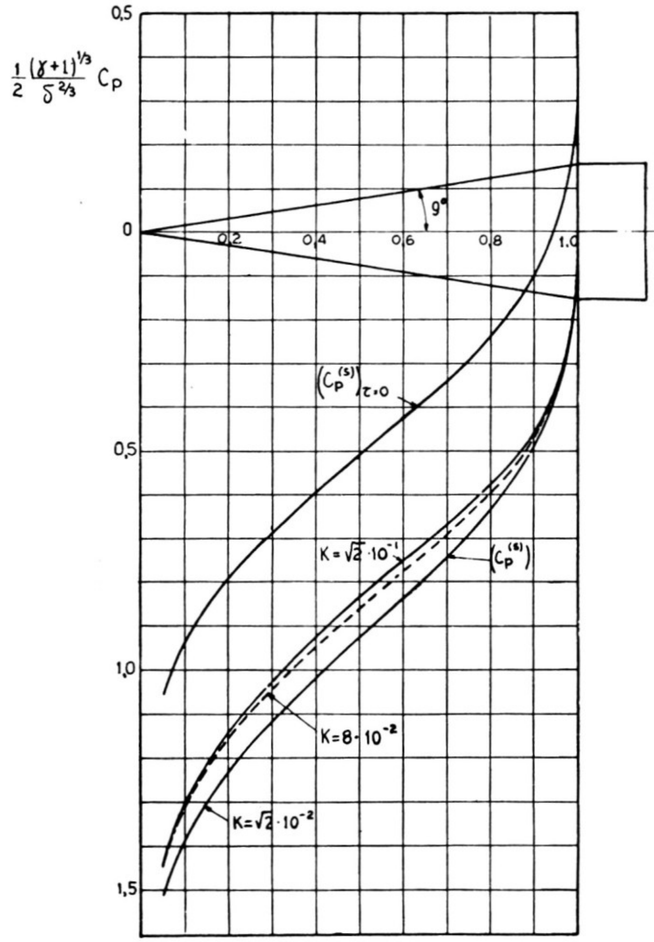

FIG. 5.

Spreiter and Alksne's method, which was extended by the Authors to the case of axial symmetric transonic flow in (21), can also be applied in the same way to the previous one to determine the acceleration influence on the pressure distribution on revolution bodies.

The equation of the motion is now

$$\frac{2K}{(\gamma+1)^{2/3}\delta^{2/3}} f^{**}_{x,\tau} + \frac{M_\infty^2 - L}{(\gamma+1)^{2/3}\delta^{2/3}} + \frac{2K\tau}{(\gamma+1)^{2/3}\delta^{2/3}} + f^{**}_x J f^{**}_{yy} - \frac{1}{y} f^{**}_y = 0 \tag{55}$$

wherein Y is now bounded to the distance $r = \frac{r}{l}$ of a general point from the axis by the same formula binding to y in the two-dimensional flow, the variable indicated with the same symbol.

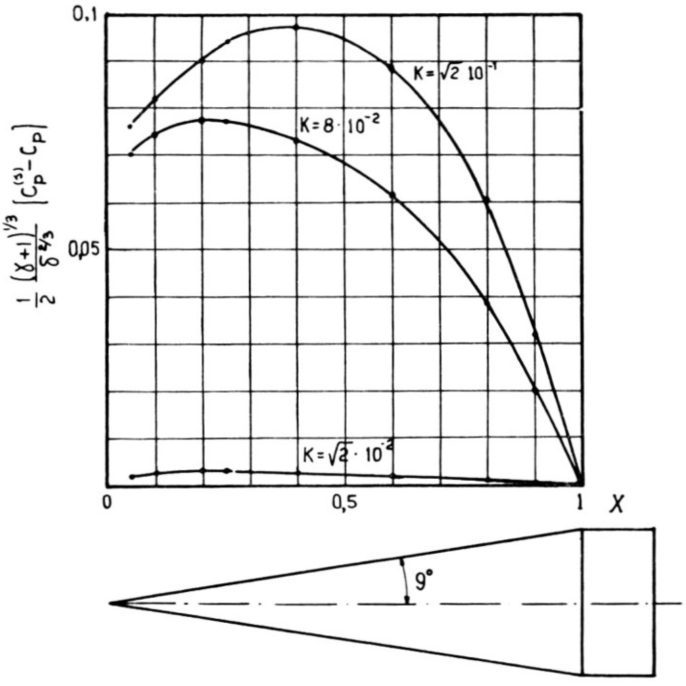

FIG. 6.

If $r = \delta g(x)$ is the equation of the meridian line of the body, the condition to be satisfied is

$$\lim_{y\to 0} (y f^{**}_y) = \frac{1}{2\pi} S' m (1 + K\tau) \tag{55'}$$

being $S' = \frac{\mathrm{d}s}{\mathrm{d}x}$; $S = \pi r^2$ = area of the cross-section; $m = \delta^{-2/3}(1+\gamma)^{1/3}$.
Applying to (55) the Laplace transformation we obtain

$$\bar{f}^{**}_{yy} - (\lambda + hp)\bar{f}^{**}_y + \frac{1}{y}\bar{f}^{**}_y = \bar{F} - h(f^{**}_x)_0 \tag{56}$$

wherein the symbols have the same meaning as in the two-dimensional flow; Eq. (55′) then gives

$$\lim_{y\to 0}(y\bar{f}_y^{**}) = \frac{1}{2\pi} S'm\left(\frac{1}{p}+\frac{K}{p^2}\right) \tag{56'}$$

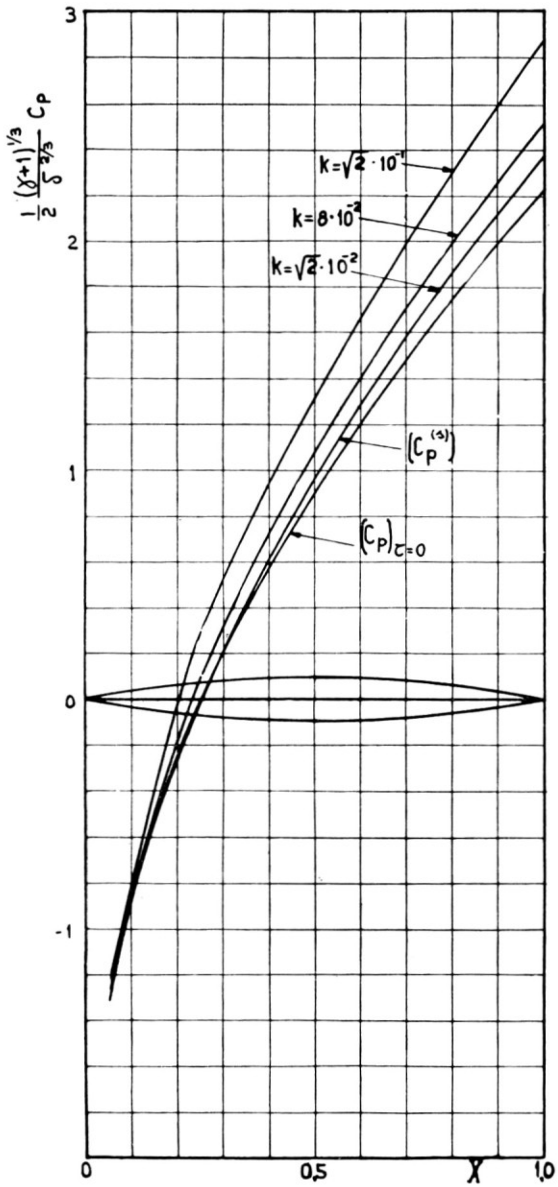

FIG. 7.

Proceeding in a way analogous to that previously indicated we obtain

$$f_x^{**} = \frac{m}{4\pi} S''(x)(1+K\tau)\log\left(e^c \frac{\lambda S}{4\pi x}\right) + \frac{m}{4\pi}(1+K\tau)\times$$

$$\times \int_0^x \frac{S''(x)-S''(\xi)}{x-\xi}\,d\xi + \frac{m}{4\pi} S''(x)\left\{-E_i\left(-\frac{\lambda\tau}{h}\right)+\right.$$

$$\left.+K\int_0^\tau -E_i\left(-\frac{\lambda\tau}{h}\right)d\tau'\right\} + \frac{1-M_\infty^2-2K\tau}{(\gamma+1)^{2/3}\delta^{2/3}} - \frac{1-M_\infty^2}{(\gamma+1)^{2/3}\delta^{/2/3}}e^{-\frac{\lambda\tau}{h}} +$$

$$+ \frac{2Kh(1-e^{-\frac{\lambda\tau}{h}})}{\lambda(\gamma+1)^{2/3}\delta^{2/3}} + e^{-\frac{\lambda\tau}{h}}(f_x^{**})_0 \tag{57}$$

for

$$\tau > \frac{hS}{4MX}(\gamma+1)^{2/3}\delta^{2/3}$$

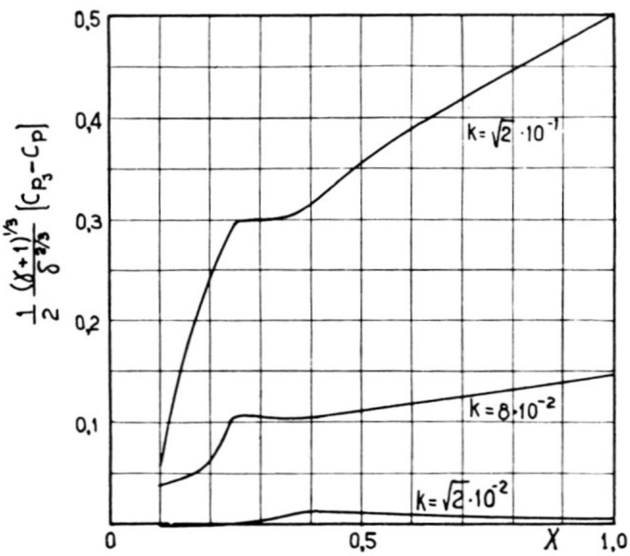

FIG. 8.

wherein C is the Euler's constant $= 0{\cdot}5772$ and $[-E_i(-e)]$ is $[-E_i(-e)]$ the integral exponential. We write Eq. (57) in the form

$$u = f_x^{**} = u^{(s)} + \Delta u \tag{58}$$

in which $u^{(s)}$ is the quasi-stationary velocity given by

$$u^{(s)} = \frac{m}{4\pi} S''(x)(1+K\tau)\log\left[e^c\left(\frac{\partial u^{(1)}}{\partial x}\right)\frac{S}{4\pi x}\right]+$$

$$+\frac{m}{4\pi}(1+K\tau)\int_0^x \frac{S''(x)-S''(\xi)}{x-\xi}\,d\xi + \frac{1-M_\infty^2-2K\tau}{(\gamma+1)^{2/3}\delta^{2/3}} \tag{58'}$$

while Δu_{∞} is the velocity variation due to the unsteadiness effects; it is given by the Eq.

$$\Delta u = \frac{m}{u\pi} S''(x)\left\{-Ei\left(-\frac{\lambda_s \tau}{h} + K\int_0^{\tau} -Ei\right) - \frac{\lambda \tau'}{h}\,\mathrm{d}\tau\right\} +$$

$$+\frac{1-M_{\infty}^2}{(\gamma+1)^{2/3}\delta^{2/3}}\, e^{-\frac{\lambda_s \tau}{h}} + \frac{\sinh(1-e^{-\frac{\lambda_s \tau}{h}})}{\lambda_s(\gamma+1)^{2/3}\delta^{2/3}} + e^{-\frac{\lambda_s \tau}{h}}(u) \qquad (58'')$$

if, analogously to that previously done, in the correction term Δu we put for λ the value $\lambda_s = \dfrac{\partial u^{(2)}}{\partial x}$. An application of the formulae (58′) and

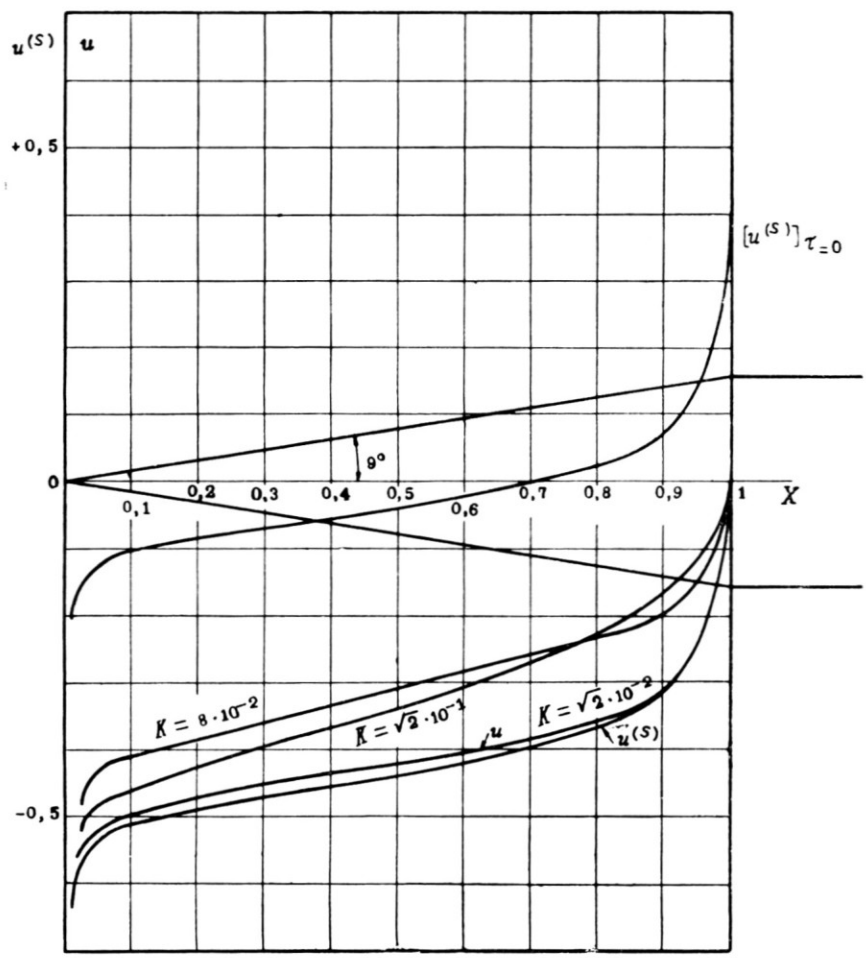

FIG. 9.

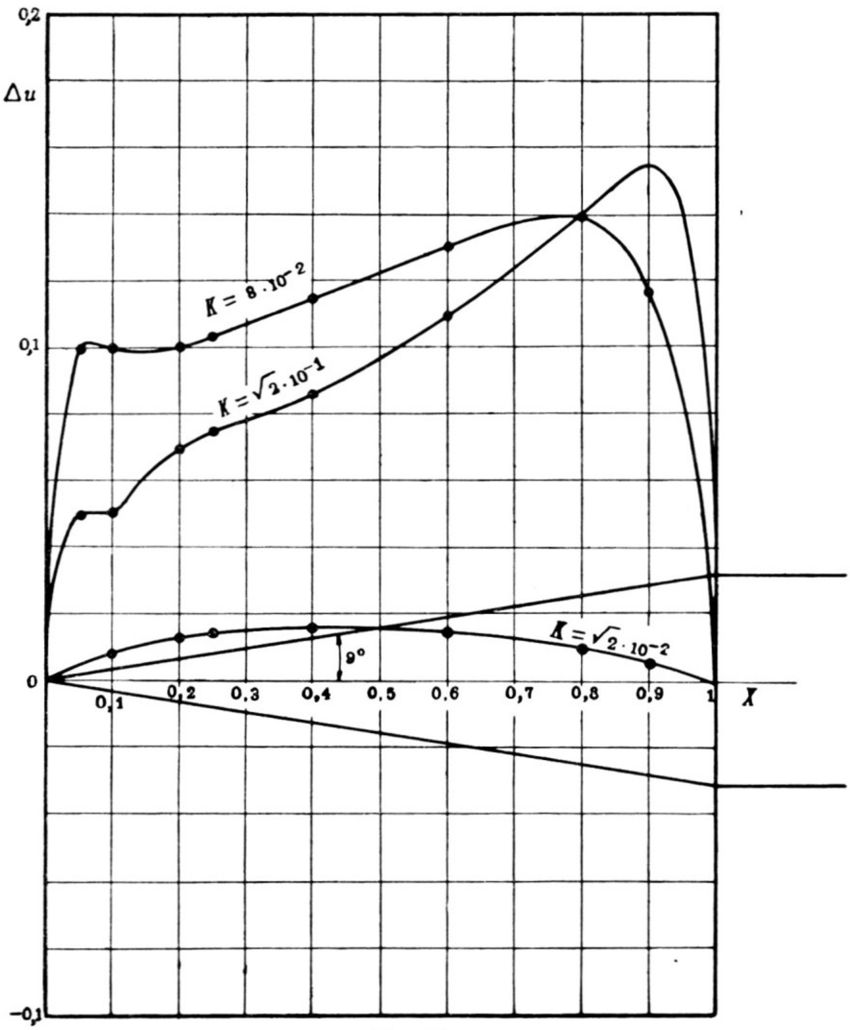

FIG. 10.

(58″) has been performed for the frustum of a cone, with a vertex semi-angle of 5°, followed by a cylindrical body and the pertinent diagrams of the pressure coefficients for $\Lambda = \sqrt{2} \cdot 10^{-2}$; $V \cdot 10^{-2}$; $\sqrt{2} \cdot 10^{-1}$ are given in Figs. 7 and 8.

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